

# Quantum Degrees of Freedom, Quantum Integrability and Entanglement Generators

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## Abstract

Dynamical algebra notion of quantum degrees of freedom is utilized to study the relation between quantum dynamical integrability and generalized entanglement. It is argued that a quantum dynamical system generates generalized entanglement by internal dynamics if and only if it is quantum non-integrable. Several examples are used to illustrate the relation.

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Quantum theory is generally considered as the fundamental theory of Physics. This means, among other things, that all physical notions that appear in it should be derived within the quantum theory itself, that is without reference to another independent physical theory. In this note we shall analyze the notion of an independent degree of freedom (IDF) as it might be (and has been) defined in quantum mechanics, and study two important issues, seemingly unrelated, where the definition of quantum mechanical degrees of freedom is relevant. i.e. the definition of entanglement and of quantum integrability.

IDF are relevant to the question of how much a state of a given system possess the properties that are typically quantum. Total level of quantum fluctuations with respect to all basic observables, which depend on the IDF of the quantum system, should be considered as a measure of quantumness of a given quantum state. It is well known that a pure state of a fixed quantum system considered in differently defined sets of IDF might show different properties of entanglement. Maximally entangled state in one set of coordinates corresponding to one set of degrees of freedom might be disentangled and posses minimal quantumness in another set of degrees of freedom.

Another property of a quantum system which depends on the definition of IDF is the notion of quantum integrability. A unique notion of the quantum dynamical integrability is not commonly accepted as it is in the classical case, and, like entanglement, depends on the definition of the IDF. A choice of IDF is possible such that any quantum system with a finite dimensional Hilbert space can be considered as a completely integrable Hamiltonian dynamical system. However, similar in the spirit and the meaning of the classical integrability is the notion of dynamical symmetry based on the systems dynamical algebra, which is well defined in quantum as well as in the classical mechanics. The notion of dynamical symmetry, and more generally the dynamical group, in quantum mechanics implies a well defined notion of independent dynamical degrees of freedom, and can be used to define quantum integrability. Although quantum systems never display qualitative properties of classical nonintegrable systems a quantum system which is quantum non-integrable according to this definition has well defined classical model which shows typical qualitative properties of chaotic classical Hamiltonian systems. On the other hand, the classical model of a quantum integrable system is completely integrable in the classical sense.

The IDF are determined by the need to describe the interactions within the system and possible interactions of the considered system and the envi-

ronment which serve to describe what can be measured in the given circumstances. For example, if the system is composed of two spatially separated subsystems than it is "natural for local observers" to choose the independent degrees of freedom to respect the spacial separation. If the spacial separation leads also to dynamical independence of the two subsystems then an arbitrary separable state of such a system remains separable in the course of the evolution, and this is an objective property of the system. In this case the two sets of entangled and separable states are dynamically separated. Thus, it could be argued that the choice of IDF in this case appears natural precisely because of the dynamical separation between entangled and separable states. This suggest that in general the quantum definition of IDF that displays the objective property of entanglement is related to the quantum dynamical properties of the system i.e. to the quantum integrability.

Our goal will be to study the importance of the definition of IDF for the relation between entanglement and quantum integrability for a quantum dynamical system in general. We shall see that in quantum mechanics the choice of degrees of freedom dictated by the dynamical structure of the system that is by the dynamical algebra and its particular subalgebras should represent also an appropriate choice of IDF for an objective and generalized treatment of entanglement. Discussion of the systems dynamical group will lead to a notion of degrees of freedom and the discussion of dynamical symmetry to the notion of quantum non-integrability and entanglement generating systems. In a nutshell, our conclusion will be that the dynamical group of the system determines its degrees of freedom and the dynamical symmetry determines if the quantum system is capable of generating entanglement.

In the next section we shall first recapitulate general definitions of independent degrees of freedom, quantum integrability and the generalized entanglement. We then establish the relation between the quantum integrability and generalized entanglement and discuss, in section 3, several examples. Dynamical algebraic definition of IDF and quantum integrability have been introduced in references [1, 2, 3]. There is no generally accepted notion of genuinely quantum integrability [4]. The definition of what is a quantum chaotic system is even less unique [5]. The most common approach, at least for lattice spin systems, is based on the generalization of the notion of thermodynamical integrability of classical spin systems[6], and is different from the one accepted here. According to the thermodynamic integrability a quantum system is called integrable if it is exactly solvable by application of the generalized Bethe ansatz or by the quantum inverse scattering method

[4]. A quantum system is nonintegrable if it has not been integrated by such methods. In what sense a quantum nonintegrable system can be considered as quantum chaotic is a matter of a debate [7]. Some quantum systems of finite number of spins whose thermodynamical limit is quantum nonintegrable, show the same spectral properties as the systems obtained by quantization of classically chaotic systems, and, furthermore, display the mixing properties that lead to expected equilibrium and non-equilibrium thermodynamical behavior [7]. The dynamics of bipartite and multipartite entanglement in such quantum chaotic systems has been studied and compared with the entanglement dynamics in quantum integrable systems [8],[9],[10],[11]. The notion of quantum integrability and nonintegrability understood in the thermodynamic sense is very different from the notion of dynamical symmetry and quantum integrability as was introduced in [1, 2, 3] and as it shall be used here.

The definition of generalized entanglement adopted here was presented in [12, 13, 14], and related definitions appear for example in [15],[16]. Here presented view of the general relation between quantum integrability, dynamics of classical approximations of quantum systems and dynamics of the generalized entanglement has not been discussed before.

## 1 General definitions

### 1.1 Dynamical algebra framework

*Kinematical degrees of freedom:*

Any quantum system with an  $N$ -dimensional Hilbert space has  $N - 1$  kinematical degrees of freedom. Its group of canonical transformations, i.e. the kinematical group, is  $U(N)$  so that any Hamiltonian, i.e. Hermitian operator, can be digitalized using some of the  $U(N)$  transformations, leading to  $N - 1$  formal integrals of motion and formal integrability. Evolution of the quantum system is equivalent to a linear symplectic flow on a symplectic manifold, which is completely integrable in the sense of classical Hamiltonian systems (please see [17] and the references therein). All pure states can be connected by some unitary transformation so that all pure states are in fact  $U(N)$  generalized coherent states. The assumption that any hermitian operator represents a measurable quantity, an observable, is actually an assumption concerning physically possible interactions with the environment,

and is not justified in many cases such as: systems of identical particles, presence of symmetries, relativistic locality etc....

### *Dynamical algebra of relevant observables*

The notion of K-freedom is too formal to be physically relevant. A particular physical system is specified, and thus distinguished from an abstract general framework, by describing what can be measured on it, i.e. by specifying the set of observables, and by expressing the interactions within the system in terms of the observables. In other words, the class of physically relevant observables should be described and the evolution should be expressed in terms of these observables. Structure of the set of observables is fixed by their algebraic relations. In quantum mechanics operators representing the physical quantities pertinent to the given system are required to realize the corresponding algebra. The algebraic relations between the observables also fix the relevant Hilbert space of the system as the space of an irreducible representation. The algebra defined in this way is called the system's dynamical algebra. Thus, a quantum system has fixed dynamical algebra. Description of a quantum system amounts to the specification of its state space, algebra of observables, or the dynamical algebra  $g$ , and the Hamiltonian which is an expression (possibly nonlinear) in terms of operators belonging to  $g$ .

In what follows we shall consider a quantum system  $(H, g, \mathcal{H})$  with a Hilbert space  $H$  which is an irrep space of the dynamical (Lie) algebra  $g$  and the Hamiltonian  $\mathcal{H}$ . The dynamical algebra  $g$  will always be a semi-simple Lie algebra, with rank  $l$  and dimension  $n$ .

## 1.2 Dynamical degrees of freedom

Dynamical degrees of freedom are fixed by the full description of the system, and are defined using the dynamical algebra. The dynamical algebra  $g$  has  $\gamma$  different chains of subalgebras:  $g \supset g_{s^l}^l \supset g_{s^{l-1}}^l \dots \supset g_1^l$ ,  $l = 1, 2, \dots, \gamma$ . Casimir operators of  $g$  and all the algebras in (any of) the subalgebra chain form the relevant complete set of commuting operators (CSCO)  $Q_j, j = 1, 2, \dots, d$ . There is  $d = l + (n - l)/2$  of these, independently of the subalgebra chain. Some of these Casimir operators are fully degenerate in the sense that they are represented by scalar operators:  $Q_i|\psi\rangle = c_i|\psi\rangle$  for every  $|\psi\rangle \in H$ . The number of non-fully degenerate operators in CSCO is  $m \leq (n - l)/2$  is chain independent but might depend on the particular irrep i.e. on the system's Hilbert space, and defines the number of IDF. The non-fully degenerate operators in a particular chain are the operators that define  $m$

IDF. The quantum system is fully specified only when 1<sup>o</sup> its Hilbert space; 2<sup>o</sup> the set of  $m$  operators representing IDF and 3<sup>o</sup> the Hamiltonian, which is a possibly nonlinear expression in terms of the dynamical algebra generators, are given.

If the dynamical algebra  $g$  of a quantum system  $C$  can be represented as a direct sum of dynamical algebras of two systems  $A$  and  $B$ , that is  $g^C = g^A \oplus g^B$ , then the tensor product of irreps of  $G^A$  and  $G^B$  is an irrep space of  $G^C$ , that is  $H^C = H^A \otimes H^B$ . If  $l_{A,B}$  and  $n_{A,B}$  are the ranks and dimensions of  $g^A$  and  $g^B$ , then in general the number of IDF of  $C$  is  $M_C = M_A + M_B$ . Thus, in the case  $g^C = g^A \oplus g^B$  the system  $C$  can be represented as a union of two systems and the number of IDF is additive. If the dynamical algebra  $g$  of the system is semisimple then it can be uniquely expressed as a direct sum of mutually commutative and orthogonal simple algebras:  $g = \oplus_k g_k$  and the Hilbert space which is an irrep space of  $g$  factors as  $H = \otimes_k H_k$ . Thus, in the case of semisimple dynamical algebra the number of IDF is additive, but the number of IDF in all the factor systems with  $g_k$  dynamical algebras need not be unity for each  $g_k$ . An example of the system when this is the case is given by a system of qubits, and shall be treated in some detail later.

However, a dynamical algebra  $g$  need not be representable as a product of dynamical Lie algebras with the number of IDF equal to one, as for example if  $g = su(3)$  or if  $A$  and  $B$  are independent fermions or bosons, even if the number of IDF of  $g$  is larger than one (for example in the  $su(3)$  case it is 2 or 3 depending on the irrep, as discussed in section 3.4).

### 1.3 Dynamical symmetry i.e. integrability

$(H, g, \mathcal{H})$  has the corresponding Lie group  $G$  as the dynamical symmetry if the Hamiltonian  $\mathcal{H}$  can be expressed in terms of the CSCO of a particular subgroup chain used to define the IDF. In this case the system has a symmetry of the subgroup chain. In particular  $H$  commutes with  $m$  non-fully degenerate operators that define the IDF.

A system  $(H, g, \mathcal{H})$  is quantum integrable by definition if it has  $G$  dynamical symmetry with respect to the subgroup chain that is used to define the IDF.

$G$  symmetry is defined as quantum integrability in analogy with complete integrability in the presence of symmetry in the case of classical Hamiltonian systems. Quantum Hamiltonian systems which do not satisfy the definition of quantum integrability are called quantum nonintegrable. It should be

stressed that the qualitative properties of the state dynamics with quantum integrable and quantum non-integrable Hamiltonians are the same. From the point of view of the Hamiltonian dynamical systems theory the state orbits are in either case regular that is periodic or quasi-periodic. Quantum non-integrable systems do not generate chaotic orbits in the system's state space (please see for example [17]). Nevertheless, the dynamical properties of orbits of the classical models (please see the next subsection) corresponding to the quantum integrable or non-integrable systems are quite different, and chaotic orbits do occur in the classical model of the quantum non-integrable systems with more than one IDF.

It should be noticed that quantum systems with one degree of freedom, unlike the one freedom classical Hamiltonian systems, need not be quantum integrable, for example if the Hamiltonian is a nonlinear expression of the algebra generators.

## 1.4 g-coherent states

Total level of quantum fluctuations in a pure state  $|\psi\rangle$  is defined as

$$\Delta(\psi) = \sum_i^n \langle \psi | L_i^2 | \psi \rangle - \langle \psi | L_i | \psi \rangle^2, \quad (1)$$

where the sum is taken over an orthonormal bases of the dynamical algebra  $g$ . It make sense to consider the quantity  $\Delta(\psi)$  as a measure of quantum-ness of the state  $\psi$ . Physical motivation for the definition of the generalized  $g$ -coherent states is that they minimize  $\Delta(\psi)$ . This is one of the important properties of the Glauber coherent states of the harmonic oscillator i.e. of the Haisenberg-Weil  $H_4$  algebra that is generalized by the  $g$ -coherent states with arbitrary  $g$ . There are several generalizations of the Glauber, i.e.  $H_4$  coherent states. Perelomov [18] and Gillmore [19] independently introduced two different generalizations based on the group-theoretical structure of the  $H_4$  coherent states. The essential ideas of both approaches are the same, the differences being in the class of Lie groups, and the corresponding available tools, and in the choice a reference state. In both approaches, the set of  $g$ -coherent states depends, besides the algebra  $g$ , also on the particular Hilbert space  $H^\Lambda$  caring the irrep  $\Lambda$  of  $g$  and on the choice of an, in principal (Perelomov), arbitrary referencee state, denoted  $|\psi_0\rangle$ . The subgroup  $S_{\psi_0}$  of  $G$  which leaves the ray corresponding to the state  $|\psi_0\rangle$  invariant is called

the stability subgroup of  $|\psi_0\rangle$ :  $h|\psi_0\rangle = |\psi_0\rangle \exp i\chi(h)$ ,  $h \in S_{\psi_0}$ . Then, for every  $g \in G$  there is a unique decomposition into the product of two elements, one from  $S_{\psi_0}$  and one from the coset  $G/S_{\psi_0}$  so that  $g|\psi_0\rangle = \Omega|\psi_0\rangle \exp i\chi(h)$ . The states of the form  $|\Lambda, \Omega\rangle = \Omega|\psi_0\rangle$  for all  $g \in G$  are the  $g$  coherent states. Thus, geometrically the set of  $g$  coherent state form a manifold with well defined Riemannian and symplectic structure.

In all explicit examples treated here the dynamical algebra  $g$  will be semisimple (or simple), which is the case studied by Gillmore. In this case there is the standard Cartan basis of  $g$ :  $\{H_i, E_\alpha, E_{-\alpha}\}$ . The irrep is characterized by the unique highest weight state  $|\Lambda, \Lambda\rangle$  (or the lowest weight state  $|\Lambda, -\Lambda\rangle$ ) which is annihilated by all  $E_\alpha$  and some  $E_{-\alpha}$ . The state  $|\Lambda, \Lambda\rangle$  is left invariant by operators in the Cartan subalgebra  $H_i$ . The set of  $g$  coherent states can be represented in the form of an action of the so called displacement operator on the reference state  $|\Lambda, \Lambda\rangle$ .

$$|\alpha\rangle = D(\alpha)|\Lambda, \Lambda\rangle = \exp\left[\sum \alpha_i E_i - h.c.\right]|\Lambda, \Lambda\rangle, \quad (2)$$

where  $\alpha_i$  are complex parameters and the sum extends over all  $E_{-\alpha}$  that do not annihilate  $|\Lambda, \Lambda\rangle$ . The stabilizer  $S_{\psi_0}$  of the reference state  $|0\rangle$  is the subgroup generated by the Cartan subalgebra of  $g$ . The complex parameters  $\alpha_i, i = 1, 2, \dots, M$  parameterize  $2M$  dimensional manifold  $G/S_{\psi_0}$ .

#### *Classical model and semi-classical dynamics*

Classical Hamiltonian dynamical system on the manifold  $G/S_{\psi_0}$  given by the Hamiltonian function  $\mathcal{H}(\alpha) = \langle \alpha | \mathcal{H} | \alpha \rangle$  is called the classical model of the quantum dynamical system  $(H, g, \mathcal{H})$ . Classical limit of the quantum system is obtained from the classical model in the limit when some relevant parameter approaches zero. If the Hamiltonian  $\mathcal{H}$  is a linear expression of the dynamical group generators then the quantum system, its classical model and its classical limit have the same dynamics. The classical model of a quantum nonintegrable system is chaotic in the sense of classical Hamiltonian dynamical systems. Dynamics of classical models of quantum nonintegrable systems have been studied for various examples in [1, 2, 3]. Relation between dynamics of entanglement and the dynamics of classical models for quantum nonintegrable pair of qubits was studied in [20].

Dynamics of the traditional semi-classical approximation of the quantum systems which are based on quantization of systems with different classical dynamics has been studied intensively[21]. It is found that a) Quantum systems obtained by quantization of classical Hamiltonian system with qual-



itatively different dynamics show different spectral properties, and qualitatively different properties of entanglement in eigenstates in different parts of spectra have been observed [10]; b) Bipartite standard entanglement, as measured by concurrence, in the wave function initially localized in qualitatively different parts of the phase space of some semi-classical approximation of the quantum system has clearly different dynamics ( for example [22, 23, 24, 25, 26, 27, 28, 29, 30]).

## 1.5 Generalized entanglement

Consider a system  $C$  such that its dynamical group  $G^C$  can be factored as a direct product  $G^C = G^A \otimes G^B$ , and its Hilbert space  $H^C$  written as the tensor product of the irreps  $H^C = H^A \otimes H^B$ . We have seen that such a system  $C$  can be viewed as a union of systems  $A$  and  $B$ . A pure state  $\psi_C$  of  $C$  is entangled by the standard definition if the reduced states  $\rho_{A,B} = Tr_{B,A}[|\psi_C\rangle\langle\psi_C|]$  are mixtures i.e. are not projectors.

g-coherent states of the system  $C$  with  $G^C = G^A \times G^B$  are products of  $g^A$  and  $g^B$  coherent states, by the construction of coherent states with the referent state  $\psi_0^C = |\psi_0^A\rangle \otimes |\psi_0^B\rangle$  and are of the form  $|\alpha^A\rangle \otimes |\alpha^B\rangle = G^A|\psi_0^A\rangle \otimes G^B|\psi_0^B\rangle$ . Reduced states  $\rho_{A,B}$  of the coherent state  $|\alpha^A\rangle \otimes |\alpha^B\rangle$  are pure and are coherent states of  $A$  and of  $B$  respectively. The coherent states of  $G^C$  are disentangled, and the reduced state of the coherent state are also coherent, and thus disentangled for the component algebras. Thus, the set of states with zero entanglement and the set of coherent states can be consistently identified. In this sense the noncoherent states do possess some entanglement in the generalized sense. If  $A$  and  $B$  are systems with only one IDF each, the previous definition assumes that the noncoherent states of  $A$  and  $B$  are entangled in the generalized sense. These states  $|\psi^A\rangle, |\psi^B\rangle$  of systems  $A$  and  $B$  with number of IDF equal to unity do have nonminimal quantumness  $\Delta(\psi)$ , and violate a Bell inequality for some set of observables [15].

In the considered case the quantumness  $\Delta(\psi^C)$  is in general larger than minimal, the minimum being achieved by states which are products of  $G^A$  and  $G^B$  coherent states. The quantumness of the state  $|\psi^C\rangle$  is here manifested in one of the two modes: a) by quantum correlations between different IDF, which is traditionally identified with entanglement, or by b) quantumness of states of systems with unit number of IDF. The definition of generalized entanglement assumes that nonminimal quantumness of noncoherent states

of systems with one IDF is equivalent to the generalized entanglement. In either the case *a*) or *b*) some Bell inequality for a convenient choice of observables is violated by a superposition of  $g$ -coherent states, that is by generalized entangled states.

Previous discussion in the case when the dynamical group satisfies  $G = G^A \otimes G^B$  is generalized by definition to the general case of the systems with dynamical groups  $G$  such that the decomposition  $G = G^A \times G^B$  does not exist. Although the system with such  $g$  dynamical algebra might have more than one IDF it can not be considered as a union of systems with smaller number of IDF. Nevertheless, the  $g$ -coherent states are defined and constructed as in the general case. The quantumness  $\Delta(\psi)$  is minimal for such coherent states and larger than minimal otherwise. States which are not  $g$ -coherent have the quantumness larger than minimal and are by definition generalized entangled or  $g$ -entangled states. Quantumness of the state  $|\psi\rangle$ :  $\Delta(\psi)$ , normalized so that it is zero for the  $g$ -coherent states can be used as a measure of  $g$ -entanglement. It was shown in [14] that it is related to the Mayer-Wallach  $Q$ -measure of multi-partite entanglement in the standard case.

Identification of  $g$ -coherent states with  $g$ -disentangled states in the case when the dynamical group does not satisfy  $G = G^A \otimes G^B$  should not be questionable. Whether a state should be considered as  $g$ -entangled whenever it is not  $g$ -coherent is a deep question with no general agreement as to the answer [15]. Following [12, 13, 14] we adopt the identification of  $g$ -entanglement with  $g$ -noncoherence. This reduces to the standard definition in the case when  $A$  and  $B$  have no entangled states and  $G = G^A \otimes G^B$ .

If this definition of  $g$ -entanglement is adopted than quantum integrability and  $g$ -entanglement are clearly related as is explained in the next subsection.

## 1.6 Entanglement generator and integrability

The system  $(H, g, \mathcal{H})$  is called an entanglement generator if it does not have  $g$  dynamical symmetry. The name is justified because an entanglement generator evolves from  $g$ -coherent into  $g$ -noncoherent states i.e. from  $g$ -disentangled into  $g$ -entangled states. Such systems produce entanglement by internal dynamics. On the other hand, if  $(H, g, \mathcal{H})$  is quantum integrable than the set of  $\{g\text{-coherent}\} \equiv \{g\text{-disentangled}\}$  is dynamically and  $G$ -invariant. In this case the Hamiltonian is not entanglement generator and such a system can be in an entangled state as a result of interaction with external systems. In other words dynamical separability can be identified with disentanglement.

This properties provide an understanding of the relation between dynamical integrability and entanglement in quantum mechanics and is the main conclusion of our discussion.

## 2 Examples

### 2.1 von-Neumann case: $u(N)$ dynamical algebra

The quantum system is described by  $N$  dimensional Hilbert space  $H^N$  and the dynamical algebra  $u(N)$ , which means that every hermitian operator on  $H^N$  has physical interpretation as a measurable quantity. Due to the normalization and global phase invariance the state space of the system is  $CP^{N-1}$  which is topologically like  $S^{2N-1}/S^1$ , and represents a  $2(N-1)$  manifold with Riemannian and symplectic structure. Geometrically, it should be natural to associate  $N-1$  IDF with this system. The same number of IDF follows from  $u(N)$  dynamical algebra. The Hilbert space is the fully symmetric irrep space of  $u(N)$  with the highest weight:  $\Lambda = (1, 0, \dots, 0)$ . The basis can be labeled by the following chain of subalgebras:  $u(N) \supset u(N-1) \dots \supset u(1)$  with the corresponding Casimir operators  $C_i^{u(k)}$ ,  $i = 1, 2 \dots k$ ,  $k = 1, 2 \dots N$  determine the irrep  $\Lambda = (1, 0, \dots, 0)$ . The  $N-1$  non-fully degenerate operators are  $C_i^{u(k)}$ ,  $i = 1, 2 \dots k$ ,  $k = 1, 2 \dots N-1$  and label the basis  $|i\rangle = |0, 0, \dots, i, \dots, 0\rangle$ ,  $i = 0, 1, 2, \dots, N-1$ . Explicitly:  $C_k^{u(k)}|i\rangle = \Theta(k - (N-i))|i\rangle$ , and  $\Theta(i)$  is the Heaviside function on  $i = 1, 2 \dots N-1$ . Thus there is  $N-1$  IDF, the same as the number of kinematical DF.

Any Hamiltonian can be diagonalized by an  $U(N)$  transformation and thus expressed as a combination of the Casimir operators. Thus any system with  $u(N)$  dynamical algebra is quantum integrable. The classical model for any quantum system with  $u(N)$  dynamical algebra is also completely integrable when considered as a classical Hamiltonian system.

Elementary excitation operators are given by:  $E_{i0}|\psi_0\rangle = |i\rangle$ ,  $i = 1, 2, \dots, N-1$  where  $|\psi_0\rangle$  is the lowest weight vector of the  $\Lambda = (1, 0, \dots, 0)$  representation, and  $U(N)$  coherent states are obtained as  $|\alpha\rangle = \exp(\sum \alpha_i E_{i1} - h.c.)|0\rangle$ . Coherent states are parameterized by the coset space  $U(N)/U(N-1) \otimes U(1)$  which is isomorphic to  $CP^{N-1}$ . We see that all states are  $U(N)$  coherent states. Thus, all states are equally and minimally quantum. The  $N-1$  degrees of freedom are disentangled in any state.

It should be noticed that since any state is  $u(N)$  coherent state the dy-

namics of the quantum system on  $CP^{N-1}$  and its classical model with the Hamiltonian function  $\langle \mathcal{H} \rangle$  on the phase space  $U(N)/U(N-1) \otimes U(1)$  are identical (and integrable) for any Hamiltonian (please see for example [17]).

A special case of the systems with  $su(N)$  dynamical group is a qubit with  $su(2)$  dynamical algebra and the Hilbert space with two complex dimensions. The number of degrees of freedom of the qubit is one, and all states, like in the general  $u(N)$  case, are coherent and equally and minimally quantum.

Systems with  $su(2)$  dynamical algebra but with the Hilbert space with  $\dim > 2$  are treated next.

## 2.2 Entanglement and quantum nonintegrability in a system with one IDF: $su(2)$ dynamical algebra with $\dim H = 2j + 1 > 2$

The two Casimir operators in the subalgebra chain:  $su(2) \supset u(1)$  are  $J^2$  and  $J_0$ . The system has only one IDF, given by the only one non-fully degenerate operator  $J_0$ . Hamiltonian which is a linear expression of the  $SU(2)$  generators is quantum integrable according to the definition (with the proper choice of the quantization axes). A system with a Hamiltonian that is a nonlinear expression of the generators is quantum nonintegrable, and as we shall see generates  $g$ -entanglement.

The  $SU(2)$  coherent states are given by:  $|\alpha\rangle = \exp((\alpha J_+ - h.c.))|0\rangle$  where  $|0\rangle$  is the unique lowest weight vector in the representation  $H^{2j+1}$  and  $J_+$  is the corresponding raising operator. States which are not coherent are more quantum in the sense that they have larger  $\Delta$  than the coherent states. According to the accepted definition such states are  $g$ -entangled.

If the Hamiltonian is a linear expression in terms of the  $su(2)$  generators, i.e. an element of the  $su(2)$  algebra then the set of coherent states is dynamically invariant. On the other hand, when the Hamiltonian is a nonlinear expression of the  $su(2)$  generators the states with different levels of quantumness are not dynamically isolated as is illustrated in fig. 1. Provided the accepted definitions of quantum nonintegrability and  $g$ -entanglement we see that nonintegrable Hamiltonians generate  $g$ -entanglement in the systems with one IDF. The data presented in fig. 1. are generated using the Hamiltonian

$$\mathcal{H} = \omega_z J_z - 2\omega_x J_x + \mu J_z^2, \quad (3)$$

which is integrable when  $\mu = 0$  and nonintegrable when  $\mu \neq 0$ .

### 2.3 Coupled spins: $su^1(2) \oplus su^2(2)$ dynamical algebra

Consider a pair of spins with the Hilbert space  $H = H^{2j_1+1} \otimes H^{2j_2+1}$  and the Hamiltonian

$$H = (1 - \mu)(J_z^1 + J_z^2) + \mu J_x^1 J_x^2, \quad (4)$$

where  $\mu \neq 1$ . The case  $\mu = 1$  is treated separately.

The dynamical group of the system is  $SU^1(2) \otimes SU^2(2)$ . The subgroup chain  $\alpha : SU^1(2) \otimes SU^2(2) \supset SO^1(2) \otimes SO^2(2)$  gives two IDF and the Casimir operators of the subgroups  $J_z^1$  and  $J_z^2$  are the observables corresponding to the two IDF.

The system is quantum integrable when  $\mu = 0$  because of the  $SU^1(2) \otimes SU^2(2)$  dynamical symmetry. If  $\mu \neq 0$  and  $\mu \neq 1$  the system is quantum nonintegrable. As was already pointed out, the orbits in the Hilbert space of the quantum integrable and nonintegrable cases belong in the same class from the point of view of the qualitative theory of dynamical systems, i.e. they are regular orbits.

Because of the definition of the dynamical group as  $SU^1(2) \otimes SU^2(2)$  the system is considered as composed of two spins.  $SU^1(2) \otimes SU^2(2)$ . Coherent states are products of the coherent states of each of the spins and are thus disentangled. If  $j_1 > 1/2$  or  $j_2 > 1/2$  then there are product states that are products of noncoherent states of the component spins. They are noncoherent but product states. These possess larger than minimal quantumness and we are considering such states as  $g$ -entangled. If the system is quantum integrable due to the symmetry  $\alpha$  the  $SU^1(2) \otimes SU^2(2)$  coherent i.e. disentangled states are dynamically invariant. Likewise, the set of noncoherent states i.e.  $g$ -entangled states is also dynamically invariant. These sets are also  $SU^1(2) \otimes SU^2(2)$  invariant. If  $\mu \neq 0$  the system is not  $SO^1(2) \otimes SO^2(2)$  quantum integrable and the sets of coherent i.e.  $g$  disentangled and noncoherent i.e.  $g$ -entangled states are not dynamically invariant. The system generates entanglement between the  $SO^1(2) \otimes SO^2(2)$  dynamical degrees of freedom (please see fig. 2).

Consider now the special case  $\mu = 1$ . It is more natural to consider the group  $SU^{1+2}(2)$  as the dynamical group of the system, with the subgroup chain  $SU^{1+2}(2) \supset SO^{1+2}(2)$ , with the  $x$ -axis as the  $SO(2)$  axis. The Hilbert space is a sum of  $j = 0, j = 1$   $SU^{1+2}(2)$  irrep spaces. The system should be considered as one degree of freedom and is quantum integrable. The level of quantumness is preserved by the evolution with the Hamiltonian for  $\mu = 1$ . Special to this case is the fact that the disen-

gled  $|1/2, -1/2 \rangle \otimes |1/2, -1/2 \rangle = |1, -1 \rangle$  and the maximally entangled  $(|1/2, -1/2 \rangle \otimes |1/2, 1/2 \rangle + |1/2, 1/2 \rangle \otimes |1/2, -1/2 \rangle)/\sqrt{2} = |1, 0 \rangle$  states are the same states in  $SU^1(2) \otimes SU^2(2)$  or  $SU^{1+2}(2) \supset SO^{1+2}(2)$  choices of the IDF, despite the fact that the number of IDF is two or one, respectively. In general, the set of  $g$ -disentangled and  $g$ -entangled states with respect to different IDF are different.

## 2.4 A simple system with entanglement: $su(3)$ dynamical algebra

The example of  $su(3)$  dynamical algebra is used to illustrate the systems with more than one IDF which nevertheless can not be considered as composed of component systems with fewer number of IDF because the Hilbert space of states does not have the corresponding tensor product structure. The example will also illustrate another important fact, namely the fact that the number of IDF might depend on the particular irrep that is carried by the system's Hilbert space.

The  $su(2)$  Lie algebra has rank 2 and dimension 8. The basic commutation relations between the generators  $E_{i,j}$ ,  $i, j = 1, 2, 3$  which are not independent are:  $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$  and can be realized in terms of bosonic creation and annihilation operators of three modes as follows:  $E_{i,j} = a_i^\dagger a_j$ ,  $i, j = 1, 2, 3$ . The eight independent hermitian generators are given by:  $X_1 = (a_1^\dagger a_1 - a_2^\dagger a_2)$ ;  $X_2 = (a_1^\dagger a_1 - a_2^\dagger a_2 - 2a_3^\dagger a_3)$ ;  $Y_k = i(a_k^\dagger a_j - a_j^\dagger a_k)$ ;  $Z_k = (a_k^\dagger a_j - a_j^\dagger a_k)$ ,  $k = 1, 2, 3$ ,  $j = k + 1 \text{ mod } 3$ . These will be used in the formula (1) for the level of  $g$ -entanglement.

In order to determine the number of IDF we need to find the number of nonfully degenerate operators in any particular chain of subalgebras. We shall use the subalgebra chain:  $su(3) \supset su(2) \oplus u(1) \supset u(1)$  with five Casimir operators usually denoted by  $C_2, C_3, Y, T^2, T_z$ .  $C_2$  and  $C_3$  are the Casimir operators of the  $su(3)$  itself,  $T^2$  and  $T_z$  are the Casimir operators of  $su(2)$  and  $u(1)$  and  $Y$  corresponds to  $u(1)$ . In the most famous application, that is in the  $SU(3)$  quark model, operators  $Y$  and  $T^2, T_z$  represent hypercharge, isospin and its  $z$  component. Thus, in general there are three nonfully degenerate operator and consequently a system with  $su(3)$  algebra has three IDF. However, the system is also characterized by its Hilbert space i.e. by a particular irrep and for some irrep all three DF might not be independent.

All irreps of the  $su(3)$  algebra can be labeled by their highest weight:

$\Lambda = \lambda_1 f_1 + \lambda_2 f_2$  where  $f_1$  and  $f_2$  are the highest weights of the two fundamental representations:  $(1, 0)$  and  $(0, 1)$ . The fully symmetric representations correspond to  $\lambda_1 = 0$  or  $\lambda_2 = 0$ . In the fully symmetric representation the operators  $T^2$  and  $Y$  are not independent and thus in this case the number of IDF is just 2. A particular Hamiltonian is quantum integrable if it is expressed in terms of  $T^2$  and  $T_z$  or  $Y, T^2$  and  $T_z$  in the two or three degrees of freedom cases. Hamiltonians used in the  $SU(3)$  quark model are integrable by construction.

The coherent states of the  $SU(3)$  dynamical group are obtained as in the general case using the highest weight vector as the reference state  $|\psi_0\rangle$ . In the general case the coherent states are parameterized by the six dimensional manifold:  $SU(3)/U(1) \otimes U(1)$  and in the case of the fully symmetric irrep with two IDF by the four dimensional  $SU(3)/U(2)$ . As usual the coherent states are of the form  $|\Lambda, \alpha\rangle = D(\alpha)|\psi_0\rangle$ . According to the adopted definition the coherent states are disentangled and quantum integrable Hamiltonian systems, like those of the quark model, can not generate entanglement. Quantum  $su(3)$  nonintegrable Hamiltonians and the generation of  $su(3)$  generalized entanglement by the system's dynamics is illustrated in the following example.

Consider the system of  $N$  particles with three possible  $N_d$ -degenerate energy levels. The following Hamiltonian for such a system is known as the Lipkin model:

$$H = \sum_{i=1}^3 \omega_i E_{ii} - \mu \sum_{i \neq j}^3 E_{i,j}^2 \quad (5)$$

where  $E_{ij}$  satisfy  $su(3)$  commutation relations. Quantum integrability and nonintegrability of this system was studied in [2], and the chaotic dynamics of the classical limit was analyzed in [31]. When  $N \leq N_d$  the Hilbert space of the system is the carrier space of the fully-symmetric irrep and the system has two IDF. If  $\mu = 0$  there is the dynamical symmetry corresponding to  $su(3) \supset su(2) \oplus u(1) \supset u(1) \times u(1)$ , the system is quantum integrable and does not generate  $g$ -entanglement. For  $\mu \neq 0 \neq \omega_i$  the system is quantum nonintegrable and does generate  $g$ -entanglement (please see fig. 3).

If  $N > N_d$  the constraints imposed on the Hilbert space by the Pauli principle become important.

## 2.5 N level system of fermions with $U(N)$ dynamical algebra

There are several dynamical algebras that can be constructed from the creation and annihilation operators of a system of identical fermions or bosons. The dynamical algebra of a system of  $k$  identical particles distributed on  $N$  levels can be chosen to be  $u(N)$ , with different Hilbert spaces of states for fermions and for bosons. In the fermion case the Hilbert space is the carrier space of the fully antisymmetric representation of  $u(N)$  denoted by  $\Lambda = (\lambda_1, \lambda_2 \dots \lambda_N) = (1, 1, \dots, 1, 0, 0 \dots 0) = (1^N, 0^{N-k})$ . The basis states of this  $d = N!/k!(N-k)!$  dimensional Hilbert space are of the form  $|n_1, n_2 \dots n_N\rangle$  with  $\sum_{i=1}^N n_i = k$ , and the number of degrees of freedom is  $N(N-k)$ .

The extremal state of the fully antisymmetric representation to be used for the construction of the coherent states is the state labeled  $|\psi_0\rangle = |1^N, 0^{N-k}\rangle$ , which is actually the ground state of the unperturbed many-fermion system. With the standard notation for the fermions creation and annihilation operators:  $a_i^\dagger, a_j$ ,  $i, j = 1, 2 \dots N$  the operators  $a_i^\dagger a_j$ ,  $i \leq k, k+1 \leq j \leq N$  generate the subgroup  $U(k) \otimes U(N-k)$  which is the stability subgroup of the reference state  $|\psi_0\rangle$ . The coherent states are of the form:  $|\Lambda, \alpha\rangle = \sum \exp(\alpha_i a_i^\dagger a_j + h.c.) |\psi_0\rangle$ , where the sum extends over  $k+1 \leq i \leq N, 1 \leq j \leq k$ . By the adopted definition, these are  $g$ -disentangled states of the fermion system, and the noncoherent states are  $g$ -entangled.

Let us consider a many fermion system with the following Hamiltonian:  $H = \sum_i^N \omega_i a_i^\dagger a_i + \mu V_{int}$ . When  $\mu = 0$ , corresponding to the noninteracting fermions the system is quantum integrable since it has the dynamical symmetry, i.e. the Hamiltonian is expressed in terms of the Casimir operators of the subgroup chain:  $U(N) \supset \dots \supset U(1) \otimes U(1) \otimes \dots \otimes U(1) \supset SO(3) \supset SO(2)$ . The coherent states are an invariant set for the evolution generated by this Hamiltonian, and such evolution does not generate  $g$ -entanglement. Of course, an analysis of the interacting systems could be to complicated but in general they are quantum nonintegrable and generate the  $g$ -entanglement.

## 3 Summary

We have used the dynamical algebra definition of independent degrees of freedom in order to establish a general relation between quantum integrability or nonintegrability and the dynamics of the generalized entanglement



(g-entanglement). Generally applicable definition of degrees of freedom of a quantum system requires specification of the system's dynamical algebra, which physically corresponds to the set of measurable observables of the system. Quantum integrability is identified with dynamical symmetry with respect to the algebra used to define the degrees of freedom. Minimal level of total quantum fluctuations is a property characteristic of the dynamical algebra generalized coherent states. States with non-minimal quantum fluctuations are here identified (following [12, 13, 14]) with the g-entangled states. With this identification, both sets of g-disentangled and g-entangled states are dynamically invariant for the quantum integrable systems. On the other hand, an orbit of the quantum nonintegrable system goes through states with zero and nonzero g-entanglement. Quantum nonintegrable systems generate g-entanglement by the internal dynamics, while quantum integrable systems can be in a g-entangled state only due to interactions with external systems. The relation between dynamical symmetry and g-entanglement is manifested also in the relation between chaotic dynamics of the quantum system's classical model and dynamical generation of g-entanglement in the quantum system. Several examples of the relation between g-entanglement and quantum nonintegrability have been discussed.

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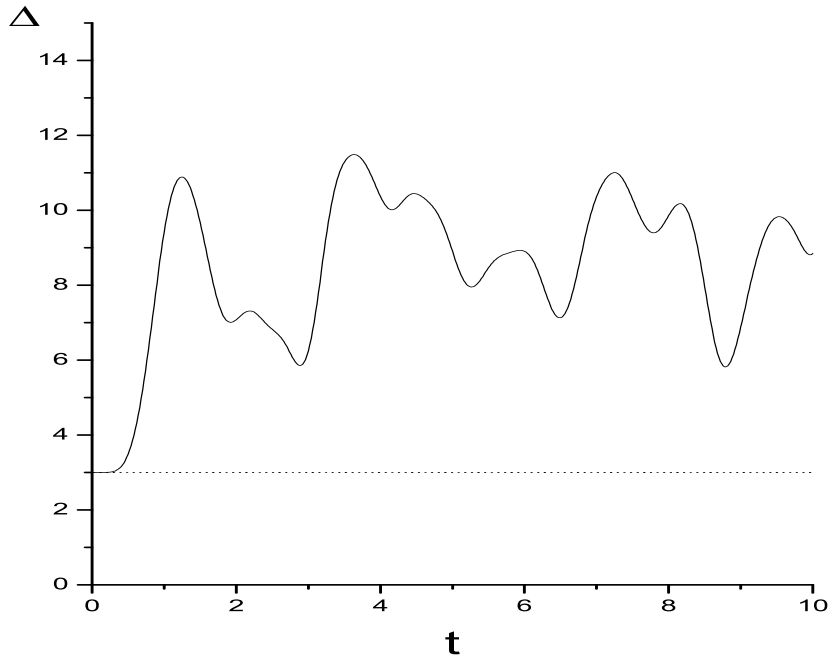


Figure 1: Illustrates dynamics of the total quantum fluctuation  $\Delta(t)$  with the Hamiltonian (3), starting from an  $SU(2)$  coherent states in  $j = 3$  irrep. Full line corresponds to the quantum nonintegrable  $\mu = 1, \omega_x = \omega_z = 1$ , and dotted to the integrable cases  $\mu = 0, \omega_x = 0, \omega_z = 1$  and  $\mu = 0, \omega_x = 1, \omega_z = 1$ .

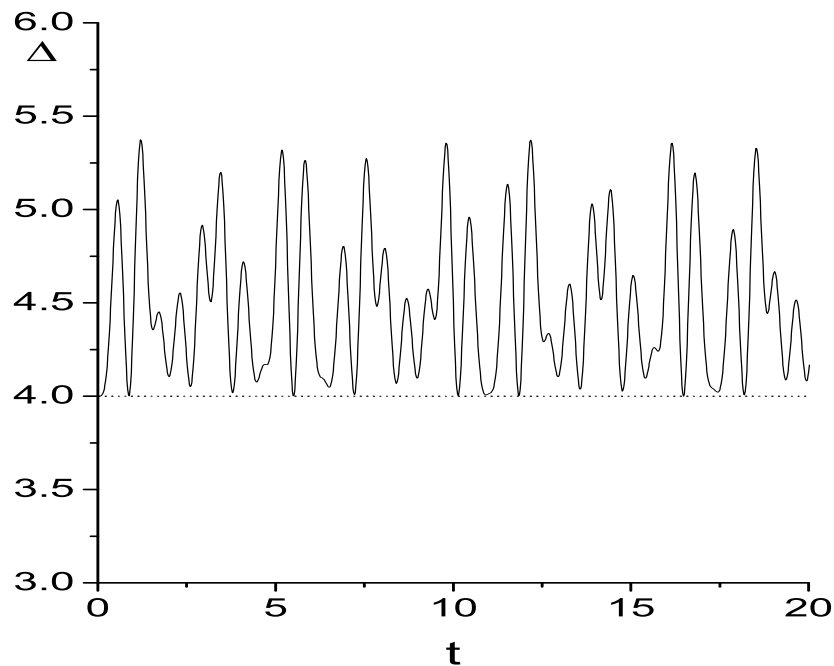


Figure 2: Illustrates dynamics of the total quantum fluctuation  $\Delta(t)$  with the Hamiltonian (4), starting from an  $SU(2) \otimes SU(2)$  coherent states in  $1/2 \otimes 1/2$  irrep. Full line corresponds to the quantum nonintegrable  $\mu \neq 0, 1$  and dotted to the integrable case  $\mu = 0$ . In the nonintegrable case the line  $\Delta(t)$  comes very close but remains larger than the initial value.

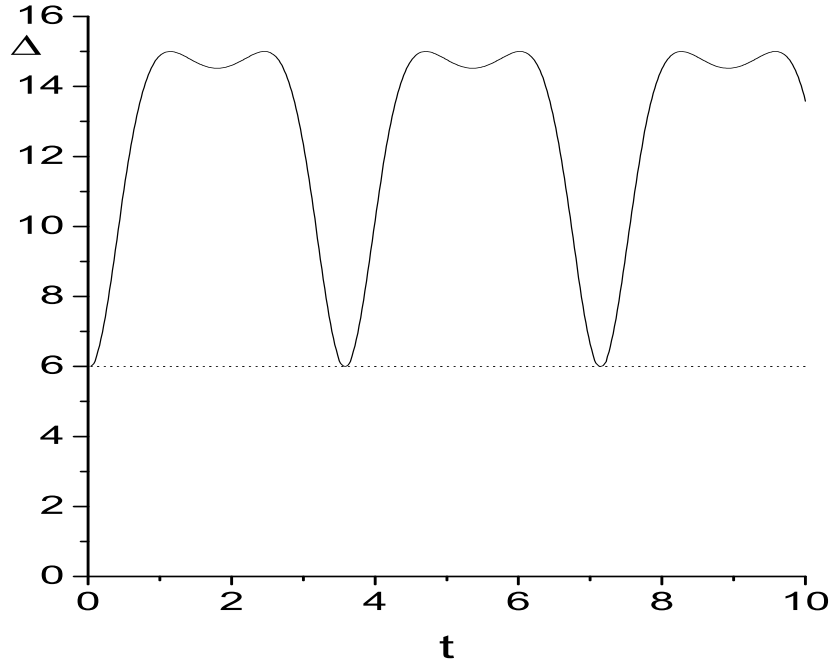


Figure 3: Illustrates dynamics of the total quantum fluctuation  $\Delta(t)$  with the Hamiltonian (5), starting from an  $SU(3)$  coherent states in the completely symmetric irrep. Full line corresponds to the quantum nonintegrable  $\mu = 1/6, \omega_i = 1$  and dotted to the integrable case  $\mu = 0, \omega_i = 1$ . In the nonintegrable case the line  $\Delta(t)$  comes very close but remains larger than the initial value.